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Nonlinear hydrodynamic and hydromagnetic spin-up driven by Ekman–Hartmann boundary layers

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Finite amplitude, impulsively started spin-up and spin-down is analysed for axially symmetric flow of a viscous, incompressible, electrically conducting fluid confined between infinite, flat, parallel, insulating boundaries. A uniform axial magnetic field is present in the initial state, but is subsequently distorted by fluid motions. The method of matched asymptotic expansions reduces the problem to a first-order, ordinary, nonlinear, integro–differential equation for the transient development of the interior angular velocity on the time scale of spin-up, as driven by quasi-steady nonlinear Ekman–Hartmann boundary layers. This two-parameter equation is solved analytically in certain limits and numerically in general. The solutions show that nonlinear non-magnetic spin-up and spin-down take longer than for linearized flow, spin-down occurring more rapidly in the early stages but requiring more time for completion than spin-up. A magnetic field promotes both spin-up and spin-down, but a weak field is relatively ineffective for spin-down yet very effective for spin-up. A strong magnetic field dominates nonlinear processes and gives identical spin-up and spin-down times, which coincide with that found from linear hydromagnetic theory.

1. Introduction

The magnetohydrodynamics of rotating fluids is a subject motivated by several important problems in geophysics and astrophysics. It is ultimately expected to provide explanations for the observed maintenance and secular variations of the geomagnetic field (e.g Hide & Roberts 1961) and clearly must also underlie much of the solar physics involved in sunspot development, the solar cycle, and, more generally, the structure of rotating magnetic stars. A recent specific question of considerable interest is whether or not the interior of the sun could still be rotating some twenty times faster than the observed surface, as proposed and supported on observational grounds by Dicke (1970). In this context, it is essential to know whether electric currents flowing in the sun's convective envelope can provide electromagnetic coupling with the radiative interior.

Many of these flows are unsteady, rotating, stratified, strongly nonlinear, fully coupled in the hydromagnetic sense and dissipative (viscously, thermally and ohmically). Given such an inclusive set of complexities, it behoves us to develop simplified but well-posed problems that contain as many interacting ingredients as possible. In this paper attention is focused on a prototype situation in rotating MHD that features all of these intricacies except stratification and thermal diffusion. Specifically, we solve the spin-up problem for a homogeneous fluid in a laterally unbounded cylindrical geometry but with strong nonlinearity and complete hydromagnetic interaction included. Direct application to the relevant physical problems is, of course, hampered by the necessary idealizations. For example, restriction to axisymmetric flow excludes dynamos (but some points of contact with dynamo theory are noted in § 6). The neglect of density stratification and compressibility is also severe, but necessary at this stage, while the artificial container geometry may be less objectionable (this is discussed further below). The work sheds light on two items of general interest that have received little prior consideration: first, the asymmetry between spin-up and spin-down that is associated with strong nonlinearity and, second, the quantitative degree to which a magnetic field tends to couple together fluid regions with different angular velocity.

The general formulation is presented in § 2. Boundary-layer and interior expansions are introduced in §§ 3 and 4, respectively. Matching and solution of the boundary-layer problem in § 5 reduces the calculation to integration of a single, nonlinear, ordinary, integro–differential equation for the evolution of the interior angular velocity. Both analytical and numerical solutions are discussed in § 6.

2. Formulation

A homogeneous, viscous, electrically conducting fluid fills a vertical circular cylinder of depth $2d$ and radius r_0. The cylinder is taken to be smooth, solid and electrically insulating (one can rationalize this as a crude simulation of the turbulent convective envelope of the sun by a fluid of infinite eddy viscosity, and infinite diffusivity for electric currents). Prior to some initial instant $t = 0$, both fluid and container are in uniform co-rotation at angular speed $\Omega_0 \geqslant 0$ about the vertical (z axis) and a uniform axial magnetic field $\mathbf{B} = B_0\,\hat{\mathbf{z}}$ is present everywhere. At time $t = 0$, the container angular speed is impulsively changed to the value $\Omega_1 \geqslant 0$ and then held fixed; no horizontal perturbations in magnetic field are allowed to develop at $|z| = \infty$.

An approximate method is used to examine, on the Ekman spin-up time scale, the ensuing flow development in which the fluid spins up (if $\Omega_1 > \Omega_0$) or spins down (if $\Omega_1 < \Omega_0$) to the new rigid rotation. Ultimate interest is in the actual time duration of the transient flow (referred to as the 'spin-up time' in either case, for convenience) and in the fluid-dynamic mechanisms responsible for the finite transition in angular velocity.

In a surprisingly wide variety of contained rotating flows of homogeneous fluids in the absence of magnetic fields, boundary layers on lateral surfaces parallel to the rotation vector exert only a secondary indirect influence on spin-up of the interior fluid, compared with Ekman layers (Greenspan 1968; Kroll & Veronis 1970). Indeed, the weakly nonlinear, non-magnetic spin-up theory developed by Greenspan & Weinbaum (1965) supports the following conclusion: on the time scale for spin-up, the temporal evolution of the angular velocity at the axis of an arbitrary smooth axially symmetric container depends only upon the geometry of the container in the polar regions; the shape of side walls has no direct effect whatever. To be precise, the same function of time describes moderately nonlinear spin-up at the axis of, for example, a sphere, an oblate or prolate spheroid, a circular cylinder, or a container formed by two infinite flat plates (i.e. a circular cylinder of infinite radius). This relative unimportance of side-wall boundary layers is also confirmed experimentally by recent work of Bien & Penner (1970).

The effect of side-wall layers on linear spin-up in the presence of an axial magnetic field can be partially deduced from the work of Ingham (1969), who studied steady flow at small magnetic Reynolds number in an insulating cylinder with differentially rotating top and bottom. In the range of parameter space which interests us here, the side-wall boundary layers were found to be (as in non-magnetic flow) of the Stewartson type, i.e. they continue to play a passive role. Consequently, we ignore side walls in the present work. A rigorous way to do this is to remove them laterally to infinity by taking the limit $r_0/d \rightarrow \infty$. The spin-up problem is henceforth treated here for the pure Ekman-type geometry originally demonstrated to be relevant by Greenspan & Howard (1963).

In non-rotating cylindrical co-ordinates (r, θ, z) the basic dependent variables are, with standard notation,

$$\mathbf{v} = (u, v, w), \quad \mathbf{A} \equiv (\rho\mu)^{-\frac{1}{2}}\mathbf{B} = (a, b, c), \quad \pi = p/\rho,$$

so that the motion is described by the MHD equations in the form

$$\partial\mathbf{v}/\partial t + \nabla(\tfrac{1}{2}\mathbf{v}.\mathbf{v}) + (\nabla\times\mathbf{v})\times\mathbf{v} = -\nabla\pi + (\nabla\times\mathbf{A})\times\mathbf{A} - \nu\nabla\times(\nabla\times\mathbf{v}), \tag{1}$$

$$\partial\mathbf{A}/\partial t = \nabla\times(\mathbf{v}\times\mathbf{A}) - \lambda\nabla\times(\nabla\times\mathbf{A}), \tag{2}$$

$$\nabla.\mathbf{v} = 0. \tag{3}$$

Since $\mathbf{A}$ is initially solenoidal, the divergence of (2) implies the redundant but useful condition

$$\nabla.\mathbf{A} = 0. \tag{4}$$

Once $\mathbf{A}$ is known, the electric current $\mathbf{j}$ follows from

$$\mathbf{j} = \mu^{-1}\nabla\times\mathbf{B} = (\rho/\mu)^{\frac{1}{2}}\nabla\times\mathbf{A}. \tag{5}$$

Prior to the impulse, the initial conditions in the fluid are

$$\mathbf{v} = \Omega_0 r\hat{\boldsymbol{\theta}}, \quad \pi = \tfrac{1}{2}\Omega_0^2 r^2, \quad \mathbf{A} = (\rho\mu)^{-\frac{1}{2}}B_0\hat{\mathbf{z}} = c_0\hat{\mathbf{z}} \quad \text{for} \quad t \leqslant 0, \tag{6}$$

where c_0 is the undisturbed Alfvén speed. Within the insulating cylinder and vacuum outside it, no currents flow so $\mathbf{A}$ is always both irrotational and solenoidal there. In the supposed absence of radial and tangential perturbations of $\mathbf{A}$ as $|z| \rightarrow \infty$ it then follows that $\mathbf{A}$ remains at most a vertical vector everywhere outside the fluid region (non-zero axial perturbations in $\mathbf{A}$ must be allowed because of field-line stretching or compression; this point is explained more fully below). Continuity of the magnetic field at the fluid–boundary interface then sets this same requirement as a boundary condition within the fluid. Consequently the boundary conditions are

$$\mathbf{v} = \Omega_1 r\hat{\boldsymbol{\theta}}, \quad \mathbf{A}\times\hat{\mathbf{z}} = 0 \quad \text{at} \quad z = \pm d, \quad \text{for} \quad t > 0, \tag{7}$$

where the origin of co-ordinates is midway between the top and bottom of the cylinder.

Equations (1)–(4), (6) and (7) contain six dimensional parameters ($\nu, \lambda, \Omega_0, \Omega_1, c_0, d$) each of which involve at most length and time. The four dimensionless parameters that can be formed are chosen as

$$E = \nu/\Omega d^2 = \text{Ekman number},$$

$$E_M = \lambda/\Omega d^2 = 1/\mu\sigma\Omega d^2 = \text{magnetic Ekman number},$$

$$\alpha = c_0/(2\lambda\Omega)^{\frac{1}{2}} = \sigma^{\frac{1}{2}}B_0/(2\rho\Omega)^{\frac{1}{2}} = \text{magnetic interaction parameter},$$

$$\epsilon = (\Omega_1 - \Omega_0)/\Omega = \text{Rossby number}.$$

In these expressions, Ω denotes the larger of Ω_0 and Ω_1, so that for spin-up (spin-down) Ω is Ω_1 (Ω_0); as a result, ϵ is positive for spin-up, negative for spin-down and $-1 \leqslant \epsilon \leqslant +1$, where $\epsilon = -1$ is the case for spin-down to rest and $\epsilon = +1$ denotes spin-up from rest, these last cases being the strongly nonlinear limits. In terms of these parameters, the magnetic Prandtl number ν/λ is E/E_M.

In the balance of this work attention is confined to the case of small viscosity but with the other dimensional parameters fixed. Since ν occurs only in E, this implies

$$E \to 0, \quad E_M = O(E^0), \quad \alpha = O(E^0), \quad -1 \leqslant \epsilon \leqslant +1.$$

The case $\epsilon \to 0$, corresponding to linearized flow, has been analysed by Loper & Benton (1970). The present work extends theirs to the complete nonlinear range, $|\epsilon| \leqslant 1$.

It is natural to scale all lengths with d, the time with the non-magnetic Ekman spin-up time and the vector $\mathbf{A}$ by its initial amplitude c_0. Together with von Kármán radial dependence, which holds without approximation in the present geometry, we set

$$\left.\begin{aligned} &\zeta = z/d, \quad \tau = (\nu\Omega/d^2)^{\frac{1}{2}}t = E^{\frac{1}{2}}\Omega t,\\ &\mathbf{v}(r,z,t) = \Omega d[(r/d)F(\zeta,\tau)\,\hat{\mathbf{r}} + (r/d)\,G(\zeta,\tau)\,\hat{\boldsymbol{\theta}} + H(\zeta,\tau)\,\hat{\mathbf{z}}],\\ &\mathbf{A}(r,z,t) = c_0[(r/d)\,K(\zeta,\tau)\,\hat{\mathbf{r}} + (r/d)\,L(\zeta,\tau)\,\hat{\boldsymbol{\theta}} + M(\zeta,\tau)\,\hat{\mathbf{z}}],\\ &\pi(r,z,t) = \tfrac{1}{2}\Omega^2r^2P(\zeta,\tau) + \Omega^2d^2Q(\zeta,\tau). \end{aligned}\right\} \tag{8}$$

With this scaling, the components of the fundamental equations (1)–(4), initial and boundary conditions (6) and (7) take on the form

$$E^{\frac{1}{2}}F_\tau - EF_{\zeta\zeta} + HF_\zeta + F^2 = -P + G^2 + 2\alpha^2E_M(MK_\zeta - 2L^2), \tag{9}$$

$$E^{\frac{1}{2}}G_\tau - EG_{\zeta\zeta} + HG_\zeta + 2FG = 2\alpha^2E_M(ML_\zeta + 2KL), \tag{10}$$

$$E^{\frac{1}{2}}H_\tau - EH_{\zeta\zeta} + HH_\zeta = -Q_\zeta, \quad P_\zeta = -4\alpha^2E_M(KK_\zeta + LL_\zeta), \quad (11),(12)$$

$$E^{\frac{1}{2}}K_\tau - E_MK_{\zeta\zeta} + HK_\zeta = MF_\zeta, \quad E^{\frac{1}{2}}L_\tau - E_ML_{\zeta\zeta} + HL_\zeta = MG_\zeta, \quad (13),(14)$$

$$E^{\frac{1}{2}}M_\tau - E_MM_{\zeta\zeta} + HM_\zeta - H_\zeta M = 0, \tag{15}$$

$$H_\zeta + 2F = 0, \quad M_\zeta + 2K = 0; \quad (16),(17)$$

at $\tau = 0$,

$$F = H = K = L = Q = 0, \quad P = \Omega_0^2/\Omega^2, \quad G = \Omega_0/\Omega, \quad M = 1; \tag{18}$$

for $\tau > 0$, at $\zeta = \pm 1$,

$$F = H = K = L = 0, \quad G = \Omega_1/\Omega. \tag{19}$$

The techniques available for solving problems of this type are severely limited. The now standard approach is to use the method of matched asymptotic expansions, which is essentially straightforward, though complicated, for the problem at hand. An alternative is the technique used by Greenspan & Weinbaum (1965) in their study of moderately nonlinear, non-magnetic spin-up. It involves an iterative approximation to the nonlinear differential equations together with asymptotic expansions for the boundary layers and inviscid interior; uniform validity in time is achieved by co-ordinate stretching in the manner of Poincaré, Lighthill and Kuo. Despite its ability to handle the complexities of arbitrary container geometry, this technique is unsuited for present purposes for several reasons. It is far from clear that extension to hydromagnetic flow is possible from a practical viewpoint. Furthermore, the time dependence (of paramount interest) is found in implicit form only. Finally, the method is sufficiently unwieldy to be incapable of covering the complete range of Rossby number (in §6 it is shown that the solutions of Greenspan & Weinbaum cannot be used for $\epsilon \geqslant \frac{4}{3}$; a secondary purpose of the present work is to remove this deficiency).

The technique of matched asymptotic expansions is effective here largely because the linearized problem is well understood (Benton & Loper 1969; Loper & Benton 1970). First note that because E_M, α and ϵ are of order E^0, the present problem is anticipated to display the same structure with respect to Ekman number as for the linear case ($\epsilon \rightarrow 0$). In particular, this implies that the spin-up time should still be proportional to $E^{-\frac{1}{2}}$, or, with present scaling, $\tau_s = O(E^0)$. On this long time scale, a diffusively growing magnetic boundary layer would grow to a thickness $(\lambda t)^{\frac{1}{2}} \sim E_M^{+\frac{1}{2}} E^{-\frac{1}{4}} d$, whereas an Alfvén wave would traverse a distance $c_0 t \sim \alpha E_M^{+\frac{1}{2}} E^{-\frac{1}{2}} d$. For α and E_M of order E^0, it is seen that magnetic diffusion or Alfvén propagation has ample time to influence all of the fluid. Since these two processes constitute the limiting situations in linearized flow (see Benton & Loper 1969, pp. 578–579) we conclude that electric currents will be important everywhere in the interior, on the time scale of spin-up. Thus, spin-up is expected to be controlled nonlinearly by quasi-steady Ekman–Hartmann boundary layers which are themselves nonlinear. A solution is sought which varies on a single time scale $\tau \sim E^0$, and two vastly different spatial scales, one being the thin scale of Ekman–Hartmann layers (refer to Gilman & Benton 1968), the other being the comparatively thick scale of the inviscid interior (essentially $2d$). The next two sections introduce asymptotic expansions of the solution functions valid in each of these two regions.

In what follows, it must be remembered that, while for linear problems it is legitimate to regard any unknown solution function as an additive superposition of an interior solution plus boundary-layer correction which vanishes (usually with exponential rapidity) as the interior is approached, in the nonlinear situation at hand, superposition fails. A more convenient approach is to regard the total solution as being given by a single term whose form changes as one's point of focus shifts from the interior to the boundary layers, or vice versa.

3. Asymptotic expansions for the Ekman–Hartmann boundary layers

By the basic reflexional symmetry about $\zeta = 0$, it suffices to examine the solution functions within just one of the boundary layers, say the lower one. There, the appropriate stretched spatial variable is

$$\eta \equiv E^{-\frac{1}{2}}(1+\zeta), \tag{20}$$

and each of the unknown functions (F, G, H, P, Q, K, L, M) takes on the form of an asymptotic power series, for example, in the boundary layer,

$$F(\zeta,\tau) = f(\eta,\tau) = f_0(\eta,\tau) + E^{\frac{1}{2}}f_1(\eta,\tau) + Ef_2(\eta,\tau) + \dots.$$

Lower-case letters are used consistently to denote the form of any function within the boundary layers. Such expansions are substituted into equations (9)–(17) and coefficients of like powers in $E^{\frac{1}{2}}$ are equated to zero. Each leading boundary-layer function ($f_0, g_0, \dots$), since it represents the total solution in the asymptotic limit $E \to 0$, is required to satisfy the appropriate exact boundary condition given in (19), while higher order terms satisfy homogeneous boundary conditions. Many terms in the asymptotic expansions are found, in this way, to be identically zero or to be functions of time only. For example, to lowest order in $E^{\frac{1}{2}}$, the continuity equation (16) becomes $h_{0\eta} = 0$ and the homogeneous boundary condition in (19) then requires that $h_0(\tau) = 0$. Similarly, it is found that k_0, l_0, p_1, q_1 and m_1 are zero, whereas m_0, p_0 and q_0 are independent of η. The equations which determine the non-zero functions of lowest order in $E^{\frac{1}{2}}$ are then given by

$$f_{0\eta\eta} - h_1 f_{0\eta} - f_0^2 + g_0^2 = p_0(\tau) - 2\alpha^2 E_M m_0(\tau)\, k_{1\eta}, \tag{21}$$

$$g_{0\eta\eta} - h_1 g_{0\eta} - 2f_0 g_0 = -2\alpha^2 E_M m_0(\tau)\, l_{1\eta}, \tag{22}$$

$$k_{1\eta\eta} = -E_M^{-1} m_0(\tau) f_{0\eta}, \tag{23}$$

$$l_{1\eta\eta} = -E_M^{-1} m_0(\tau)\, g_{0\eta}, \tag{24}$$

$$h_{1\eta} = -2f_0. \tag{25}$$

These equations show that the boundary-layer problem is inertially and electromagnetically nonlinear. However, since time derivatives are absent, the boundary layers are quasi-steady and time plays the role of a passive parameter. Equations (23) and (24) reveal that induction of horizontal fields by shear-induced tipping of the axial field is balanced by magnetic diffusion within these boundary layers. The boundary-layer analysis places no dynamic restriction on the leading contribution to the axial magnetic field, m_0, other than that it be spatially uniform within the boundary layer; this suggests that $m_0(\tau)$ will be determined by the interior dynamics and is not an unknown in the boundary-layer problem. Finally, note that although the magnetic Ekman number occurs separately in, for example, (21) and (23) it disappears when (23) is substituted into the η derivative of (21).

The boundary-layer equations can be compacted by introducing two complex functions (denoted by a tilde) of the real variables $(\eta;\tau)$, where the semicolon emphasizes the passive nature of τ within the boundary layers:

$$\tilde{f}(\eta;\tau) \equiv f_0(\eta;\tau) + ig_0(\eta;\tau); \quad \tilde{k}(\eta;\tau) = k_1(\eta;\tau) + il_1(\eta;\tau). \tag{26}$$

With this notation, the boundary-layer problem can be expressed as

$$\tilde{f}_{\eta\eta\eta} - h_1\tilde{f}_{\eta\eta} - [h_{1\eta} + 2\tilde{f} + 2\alpha^2 m_0^2(\tau)]\tilde{f}_\eta = 0, \tag{27}$$

$$h_1(\eta;\tau) = -\int_0^\eta [\tilde{f}(\xi;\tau) + \tilde{f}^*(\xi;\tau)]\,d\xi, \tag{28}$$

$$\tilde{k}_{\eta\eta} = -E_M^{-\frac{1}{2}} m_0(\tau)\tilde{f}_\eta, \tag{29}$$

$$\tilde{f}(0;\tau) = i\,\Omega_1/\Omega, \quad \tilde{k}\,(0;\tau) = 0, \tag{30}$$

where the asterisk in (28) signifies a complex conjugate. In this form, the cubic nonlinearity of the electromagnetic body force is to be noted. Clearly, to complete the specification of the boundary-layer problem, two more conditions on $\tilde{f}$ and one more on $\tilde{k}$ are required. These are supplied by matching the boundary-layer expansions to interior expansions. The problem given by (27), (28) and (30) reduces properly, in the non-magnetic limit $\alpha \to 0$, to one treated by Fettis (1955).

4. Asymptotic expansions for the inviscid interior

When attention is shifted from the boundary layers to the inviscid interior, then the spatial scale for variation of any solution function changes to the depth of the interior fluid, which is of order $2d$ when the boundary layers are thin. Thus, the correct variable for outer expansions is ζ and hence all spatial derivatives are weaker (in powers of $E^{\frac{1}{2}}$) than within the boundary layers.

Since H, K and L are zero to order $E^{\frac{1}{2}}$ within the boundary layers, they must (from considerations of smooth matching) remain zero to that same order in the interior, i.e. if capital letters are retained to denote any function evaluated within the interior but with a subscript to indicate the power of $E^{\frac{1}{2}}$, we have

$$H_0 = K_0 = L_0 = 0.$$

From (11) and (12) this in turn implies that $Q_{0\zeta} = Q_{1\zeta} = P_{0\zeta} = P_{1\zeta} = 0$ and the values which merge smoothly into the boundary-layer values are clearly

$$P_0 = P_0(\tau) = p_0(\tau), \quad P_1 = 0, \quad Q_0 = Q_0(\tau) = q_0(\tau) \quad \text{and} \quad Q_1 = 0.$$

Since $K_0 = 0$, equation (17) plus matching with the boundary layer show that $M_0 = M_0(\tau) = m_0(\tau)$. Because m_1 is zero within the boundary layer, it then must remain zero in the interior. Hence, from (17), $K_1 = 0$. These arguments motivate interior expansions of the form

$$\left.\begin{aligned}
F(\zeta,\tau) &= F_0(\zeta,\tau) + E^{\frac{1}{2}}F_1(\zeta,\tau) + \ldots,\\
G(\zeta,\tau) &= G_0(\zeta,\tau) + E^{\frac{1}{2}}G_1(\zeta,\tau) + \ldots,\\
H(\zeta,\tau) &= E^{\frac{1}{2}}H_1(\zeta,\tau) + \ldots,\\
P(\zeta,\tau) &= P_0(\tau) + EP_2(\zeta,\tau) + \ldots,\\
Q(\zeta,\tau) &= Q_0(\tau) + EQ_2(\zeta,\tau) + \ldots,\\
K(\zeta,\tau) &= EK_2(\zeta,\tau) + \ldots,\\
L(\zeta,\tau) &= E^{\frac{1}{2}}L_1(\zeta,\tau) + \ldots,\\
M(\zeta,\tau) &= M_0(\tau) + EM_2(\zeta,\tau) + \ldots.
\end{aligned}\right\} \tag{31}$$

These are now substituted into (9), (10) and (13)–(16) and coefficients of the terms of order E^0 and $E^{\frac{1}{2}}$ are equated to zero. Each equation in this set (except for (15), which is already satisfied to order E^0) produces two new relations as follows:

$$F_0^2 = -P_0(\tau) + G_0^2, \tag{32}$$

$$F_{0\tau} + H_1 F_{0\zeta} + 2F_0 F_1 = 2G_0 G_1, \quad F_0 G_0 = 0, \tag{33),(34}$$

$$G_{0\tau} + H_1 G_{0\zeta} + 2F_0 G_1 + 2F_1 G_0 = 2\alpha^2 E_M M_0(\tau) L_{1\zeta}, \tag{35}$$

$$M_0 F_{0\zeta} = 0, \quad M_0 F_{1\zeta} = 0, \quad M_0 G_{0\zeta} = 0, \tag{36),(37),(38}$$

$$M_0 G_{1\zeta} = -E_M L_{1\zeta\zeta}, \quad M_{0\tau} - H_{1\zeta} M_0 = 0, \tag{39),(40}$$

$$F_0 = 0, \quad H_{1\zeta} = -2F_1. \tag{41),(42}$$

Since (41) implies that there is no radial motion in the interior to leading order, (32) and (38) show that the azimuthal flow evolves in time as a geostrophic rigid rotation which satisfies the Taylor–Proudman constraint (as for non-magnetic flow, see Greenspan & Weinbaum 1965):

$$G_0 = G_0(\tau). \tag{43}$$

With M_0 a function of τ only, (40) implies that $H_{1\zeta}$ (and hence F_1, by (42)) is a function of τ only. Because $F_0 = 0$, equation (33) shows that $G_1 = 0$, whence G_0 is the azimuthal flow to order E. By (39) then $L_{1\zeta}$ is a function of τ only which is also consistent with (35).

The main equation of interest, namely that for the temporal evolution of the interior angular velocity (35), can be written as

$$dG_0(\tau)/d\tau = -2F_1(\tau)\, G_0(\tau) + 2\alpha^2 E_M M_0(\tau)\, L_{1\zeta}(\tau). \tag{44}$$

The interpretation of this equation is aided by finding the electric current in terms of the scaling introduced. Substitution of (8) into (5) gives

$$\mathbf{j} = \frac{B_0}{\mu d}\left[-\frac{r}{d} L_\zeta \hat{\mathbf{r}} + \frac{r}{d} K_\zeta \hat{\boldsymbol{\theta}} + 2L\,\hat{\mathbf{z}}\right], \tag{45}$$

so the azimuthal field component L corresponds to axial current, and its vertical derivative (the last factor in (44)) is a radial current. Clearly, spin-up or spin-down results from two distinct mechanisms in (44). An azimuthal electromagnetic body torque arises from the interaction of radial currents ($L_{1\zeta}$) crossing the axial magnetic field (M_0). Simultaneously, or in the absence of magnetic fields ($\alpha = 0$), spin-up or spin-down occurs because interior fluid parcels conserve their angular momentum as they drift radially under the influence of secondary flow induced by Ekman boundary layers, and vorticity is added by the stretching of vortex lines (from (42), the radial flow factor $-2F_1$ in (44), is the same as $H_{1\zeta}$, and this measures the rate of extension of a vertical line element). Note further that both of these physical mechanisms are actually nonlinear, although they can be represented within a linear theory.

Finding the solution for $G_0(\tau)$ first requires determining how F_1, $L_{1\zeta}$ and M_0 depend functionally on G_0. To the order we have worked, time derivatives of

F_1 and $L_{1\xi}$ do not occur in the interior equations, so the interior radial flow and radial current simply respond to the boundary-layer forcing (in essence, the inviscid interior is connected to and driven by an Ekman–Hartmann pump). However, the axial field M_0 has its own interior dynamics (as anticipated previously) and, by virtue of (40) and (42), is given by

$$dM_0(\tau)/d\tau = H_{1\xi}(\tau)\, M_0(\tau) = -2F_1(\tau)\, M_0(\tau). \tag{46}$$

The work of Benton & Loper (1969) reveals that a (linear) Ekman–Hartmann boundary layer requires at most of order 2 radians of boundary rotation to develop, starting from rest. With the present scaling, this corresponds to a value of $\tau = 2E^{\frac{1}{2}}$. For consistency with the asymptotic limit $E \to 0$, this boundary-layer formation time is ignored, and an initial condition for (46) is that given in (18). The relevant solution for $M_0(\tau)$ is then

$$M_0(\tau) = \exp\left[-2\int_0^\tau F_1(\tau')\, d\tau'\right]. \tag{47}$$

The transcendental nonlinearity evident here is a feature of major interest. Equation (40) reveals that, on the long time scale for spin-up, the term describing magnetic diffusion of vertical flux is identically zero within the interior, so the secondary flow simply convects axial magnetic field as though that component were frozen into the fluid. Just as the Ekman suction (or blowing) stretches (or contracts) axial vortex lines, so it also stretches (or contracts) axial magnetic field lines. Consequently, the basic axial field amplifies exponentially during spin-up because $F_1 < 0$ and decays during spin-down because then $F_1 > 0$. This nonlinear change in axial field is communicated, unabated, across the boundary layers (recall that $\partial m_0/\partial\eta = 0$) and, by continuity of field, all the way to spatial infinity. This explains why only horizontal perturbations in the field can be required to vanish at $|z| = \infty$. By way of contrast, in linear hydromagnetic spin-up (Loper & Benton 1970), the interior axial magnetic field is treated as constant (as is the factor G_0 in the hydrodynamic spin-up mechanism given by the term $-2F_1G_0$ in (44)).

When (47) is substituted into (44), we obtain the interior problem in the form

$$\frac{dG_0(\tau)}{d\tau} = -2F_1G_0 + 2\alpha^2E_M L_{1\xi} \exp\left[-2\int_0^\tau F_1(\tau')\, d\tau'\right], \tag{48}$$

subject to the initial condition (from 18)

$$G_0(0) = \Omega_0/\Omega. \tag{49}$$

5. Matching of the expansions and solution of the boundary-layer problem

Solution of the interior equation (48) necessitates relating the radial flow F_1 and radial current $L_{1\xi}$ to the angular velocity G_0. In view of the solenoidal nature of both velocity and electric current, this is tantamount to finding the functional relationship between vertical velocity, vertical current and angular velocity (or vertical vorticity). The vertical velocity and current are found simply in the interior, since they obey the interior equations (from (39) and (42))

$$H_{1\zeta\zeta} = 0, \quad L_{1\zeta\zeta} = 0. \tag{50}$$

By reflexional symmetry there can be neither mass flux nor charge flux (i.e. current) crossing the mid-plane $\zeta = 0$, so one boundary condition for each of these equations is

$$H_1(0,\tau) = 0, \quad L_1(0,\tau) = 0. \tag{51}$$

The other conditions arise from matching. The vertical velocity and electric current established by the Ekman–Hartmann pump at the outer edge of the boundary layer ($\eta \to \infty$) must be the same as that seen from the interior as $\zeta \to -1$, where the boundary-layer thickness, being of order $E^{\frac{1}{2}}$ (Benton & Loper 1969), is ignored in the limit $E \to 0$. Hence,

$$H_1(-1,\tau) = h_1(\infty;\tau), \quad L_1(-1,\tau) = l_1(\infty;\tau). \tag{52}$$

The solutions of (50)–(52) are

$$H_1(\zeta,\tau) = -h_1(\infty;\tau)\,\zeta, \quad L_1(\zeta,\tau) = -l_1(\infty;\tau)\,\zeta. \tag{53}$$

Consequently,

$$F_1(\tau) = -\tfrac{1}{2}H_{1\zeta} = \tfrac{1}{2}h_1(\infty;\tau), \quad L_{1\zeta}(\tau) = -l_1(\infty;\tau), \tag{54}$$

so that (48) becomes

$$\frac{dG_0(\tau)}{d\tau} = -h_1(\infty;\tau)\,G_0(\tau) - 2\alpha^2 E_M\, l_1(\infty;\tau)\exp\left[-\int_0^\tau h_1(\infty;\tau')\,d\tau'\right]. \tag{55}$$

This serves again to emphasize how the interior angular velocity is driven by the Ekman–Hartmann pump through its induced axial velocity $h_1(\infty;\tau)$ and axial electric current $l_1(\infty;\tau)$.

It remains to solve the boundary-layer problem (27)–(30) in order to evaluate $h_1(\infty;\tau)$ and $l_1(\infty;\tau)$ as functions of $G_0(\tau)$ for insertion into (55). Two outer boundary conditions for $\tilde{f}$ and one for $\tilde{k}$ still need to be specified. Clearly, all the non-vanishing order E^0 quantities in the interior (G_0, P_0, Q_0, M_0) must coincide with the values found at the outer edge of the boundary layers, so

$$\tilde{f}(\infty;\tau) = iG_0(\tau), \quad m_0(\tau) = M_0(\tau). \tag{56}$$

Also, since the interior flow has no order E^0 vertical variation,

$$\tilde{f}_\eta(\infty;\tau) = 0. \tag{57}$$

For $\tilde{k}$ it is sufficient to require boundedness,

$$|\tilde{k}(\infty;\tau)| < \infty. \tag{58}$$

The complete problem to be solved is now expressible in sequential form. From (27), (28), (30), (47), (54), (56) and (57) the problem for $\tilde{f}$ is given by

$$\left.\begin{aligned} &\tilde{f}_{\eta\eta\eta} - h_1\tilde{f}_{\eta\eta} - \left\{h_{1\eta} + 2\tilde{f} + 2\alpha^2\exp\left[-2\int_0^\tau h_1(\infty;\tau')\,d\tau'\right]\right\}\tilde{f}_\eta = 0, \\ &h_1(\eta;\tau) = -\int_0^\eta [\tilde{f}(\xi;\tau) + \tilde{f}^*(\xi;\tau)]\,d\xi, \\ &\tilde{f}(0;\tau) = i\Omega_1/\Omega, \quad \tilde{f}(\infty;\tau) = iG_0(\tau), \quad \tilde{f}_\eta(\infty;\tau) = 0. \end{aligned}\right\} \tag{59}$$

From the defining equation (26), this in turn gives $f_0(\eta;\tau)$ and $g_0(\eta;\tau)$ in terms of $G_0(\tau)$. The Ekman pumping function $h_1(\infty;\tau)$ is also determined, by the middle equation of (59). Once $\tilde{f}$ is known, then k_1 and l_1 are found via (26) and the following problem (which arises from (29), (30), (47), (54), (56) and (58))

$$\left.\begin{aligned} \tilde{k}_{\eta\eta} &= -E_M^{-1}\tilde{f}_\eta \exp\left[-\int_0^\tau h_1(\infty;\tau')\,d\tau'\right], \\ \tilde{k}(0;\tau) &= 0, \quad |\tilde{k}(\infty;\tau)| < \infty. \end{aligned}\right\} \tag{60}$$

The solution of (60) determines the Hartmann current function $l_1(\infty;\tau)$. The final step requires solving the interior equation (55) subject to (49).

There are three distinct sources of nonlinearity in (55). From (59) and (60), it is clear that $\tilde{f}$ and hence $\tilde{k}$ depend nonlinearly on G_0. This shows up in (55) as nonlinear dependence of the Ekman pumping function $h_1(\infty;\tau)$ and Hartmann current function $l_1(\infty;\tau)$ on $G_0(\tau)$. This is referred to as the 'boundary-layer nonlinearity'. In effect, the interior is connected to a nonlinear Ekman–Hartmann pump. A second nonlinearity is the 'inertial nonlinearity' represented by the term $G_0(\tau)$ on the right-hand side of (55), which is approximated by the constant initial value $G_0(0) \doteq 1$ in linear theory. Finally, the transcendental 'electromagnetic nonlinearity', the exponential term in (55), expresses temporal changes in the basic axial field forced by the Ekman–Hartmann secondary flow.

The question now arises of how to handle the various nonlinearities. Undoubtedly, the effects of the inertial and electromagnetic nonlinearities are least well known so they are treated exactly in what follows. However, it has been found expedient and sufficiently accurate to approximate the boundary-layer nonlinearity by what is, in effect, a regular perturbation expansion in Rossby number truncated after the terms quadratic in that parameter. A first rationale for this step rests on the observation that existing solutions for steady nonlinear Ekman boundary layers (summarized, for example, in Greenspan 1968, p. 140) reveal only modest departures of Ekman pumping from the predictions of linear theory unless the nonlinearity is strong (say $|\epsilon| \geqslant 0{\cdot}5$). The corresponding situation for steady nonlinear Ekman–Hartmann layers is clarified by recent work of Benton & Chow (1972). It is found there that Ekman pumping and Hartmann current as predicted by a quadratic Rossby number expansion can each be in significant disagreement with exact numerical solutions. However, for spin-up or spin-down, the particular combination of these quantities that occurs (developed below) is predicted very well indeed by the cruder technique (because the discrepancies in the individual predictions tend to compensate each other).

It should be noted that both the inertial nonlinearity and the electromagnetic nonlinearity are initially zero but develop as time progresses (the former because initially the interior angular velocity has not yet changed, the latter because field lines have not yet been stretched or contracted). In contrast, the boundary-layer nonlinearity is strongest initially, because then the angular velocity difference across the layers is largest. This suggests that the relative importance of boundary-layer nonlinearity can be studied in isolation by defining the strength S of a nonlinear Ekman–Hartmann pump as the *initial* angular acceleration it

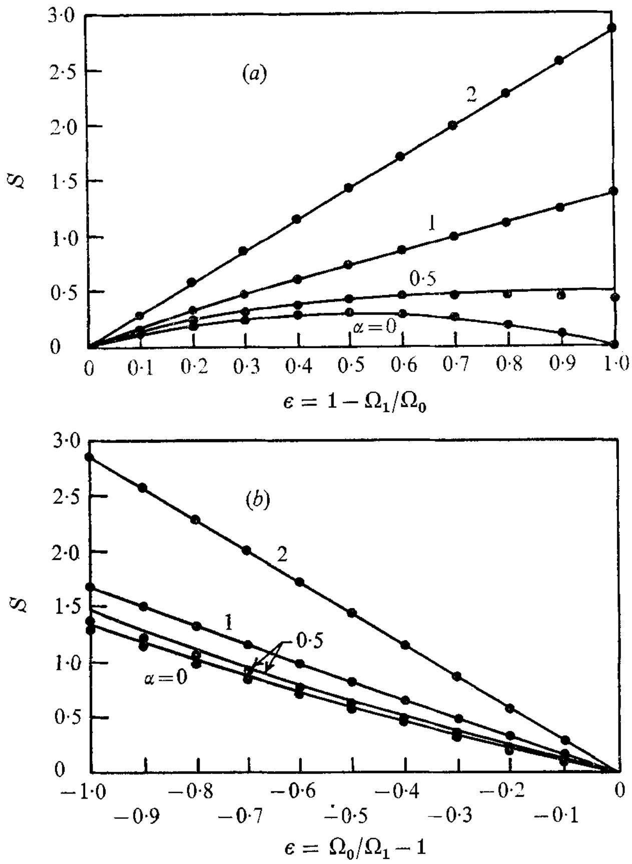

FIGURE 1. Strength of a steady Ekman–Hartmann pump for (*a*) spin-up and (*b*) spin-down (after Benton & Chow 1972). ——, exact numerical solution; ●, quadratic Rossby number expansion.

produces in the spin-up problem. Evaluation of (55) at $\tau = 0$ with the given initial condition on G_0 yields

$$S \equiv \left.\frac{dG_0(\tau)}{d\tau}\right|_{\tau=0} = \begin{cases} -h_1(\infty;0)\,(1-\epsilon) - 2\alpha^2 E_M l_1(\infty;0) & \text{for spin-up,} \\ -h_1(\infty;0) - 2\alpha^2 E_M l_1(\infty,0) & \text{for spin-down.} \end{cases} \tag{61}$$

Figures 1(*a*) and (*b*), based on work by Benton & Chow (1972), show that S, as measured by these expressions, is very well approximated for all α and all $|\epsilon| \leqslant 1$ by a two-term expansion in Rossby number, thereby justifying the method adopted below. In these figures, the solid curves are the results of full numerical integration whereas the solid points result from the truncated expansion.

Because the initial condition for spin-up differs from that for spin-down, the analysis must be bifurcated at this point.

5.1. *Boundary-layer problem for spin-up*

When the fluid is spinning up, $\Omega = \Omega_1$ and therefore $\epsilon = 1-\Omega_0/\Omega_1 \geqslant 0$. Also, for this case $G_0(0) = \Omega_0/\Omega_1 = 1-\epsilon$ and $G_0(\infty) = 1$ so the following equality holds: $0 \leqslant 1-G_0(\tau) \leqslant \epsilon \leqslant 1$. The quantity $1-G_0(\tau)$ measures the instantaneous deviation of the interior angular velocity (or vorticity) from its final value so it plays the role of an instantaneous Rossby number. The task at hand requires finding from (59) how $\tilde{f}$ and hence h_1 depend upon this essentially passive parameter. We proceed as if ϵ, and thus $1-G_0(\tau)$, were truly small, i.e. we first content ourselves with an approximate solution linear in $1-G_0(\tau)$. Expanding about conditions at the boundary, let

$$\tilde{f}(\eta;\tau) = i - i[1-G_0(\tau)]\tilde{f}^{(1)}(\eta).$$

The middle equation in (59) then becomes

$$h_1(\eta;\tau) = i[1-G_0(\tau)]\int_0^\eta [\tilde{f}^{(1)}(\xi) - \tilde{f}^{(1)*}(\xi)]\,d\xi,$$

so that the quantity in the exponential of the first equation in (59) is

$$-2\int_0^\tau h_1(\infty;\tau')\,d\tau' = -2i\int_0^\tau [1-G_0(\tau')]\,d\tau'\int_0^\infty [\tilde{f}^{(1)}(\xi) - \tilde{f}^{(1)*}(\xi)]\,d\xi.$$

Since $1-G_0$ is regarded as small and the spin-up time is expected to be of order $\tau_s \sim 1$, the first integral factor here is small. Hence, the exponential is expandable in rapidly convergent power series. The terms linear in $1-G_0(\tau)$ in the first equation of (59) then lead to the following problem for $\tilde{f}^{(1)}(\eta)$:

$$\tilde{f}^{(1)}_{\eta\eta\eta} - (2\alpha^2+2i)\tilde{f}^{(1)}_\eta = 0, \quad \tilde{f}^{(1)}(0) = \tilde{f}^{(1)}_\eta(\infty) = 0, \quad \tilde{f}^{(1)}(\infty) = 1.$$

This is just the problem for a steady linear Ekman–Hartmann boundary layer (Gilman & Benton 1968), the solution being

$$\tilde{f}^{(1)}(\eta) = 1 - e^{-(\beta+i\gamma)\eta},$$

where $$\beta = [\alpha^2 + (1+\alpha^4)^{\frac{1}{2}}]^{\frac{1}{2}}, \quad \gamma = \beta^{-1} = [-\alpha^2 + (1+\alpha^4)^{\frac{1}{2}}]^{\frac{1}{2}}.$$

Thus, to first order in $1-G_0(\tau)$, the boundary layer is just a linear Ekman–Hartmann layer with passive time-varying azimuthal flow outside and constant axial magnetic field.

The first correction for nonlinear effects is determined by setting

$$\tilde{f}(\eta;\tau) = i - i[1-G_0(\tau)]\,[1-e^{-(\beta+i\gamma)\eta}] - i\tilde{f}^{(2)}(\eta;\tau),$$

where $|\tilde{f}^{(2)}(\eta;\tau)|$ tends to zero faster than ϵ or $1-G_0$. This is now substituted into (59); the exponential term is again expanded in power series, and a linear equation for $\tilde{f}^{(2)}$ is developed. The solution of this equation is straightforward, though laborious, the result being the following approximation for $\tilde{f}$, which is exact to second order in $1-G_0$:

$$\tilde{f}(\eta;\tau) = i - i[1-G_0(\tau)]\,[1-e^{-(\beta+i\gamma)\eta}] - i[1-G_0(\tau)]^2\,[(A\eta+B)\,e^{-(\beta+i\gamma)\eta} - B\,e^{-2\beta\eta}]$$
$$- i[1-G_0(\tau)]\left[\int_0^\tau [1-G_0(\tau')]\,d\tau'\right] C\eta\, e^{-(\beta+i\gamma)\eta}, \quad (62)$$

where $$A = -i\frac{\beta}{\beta^2+\gamma^2}, \quad B = i\frac{(\beta+i\gamma^3)\,(3\beta-i\gamma)}{(\beta^2+\gamma^2)\,(9\beta^2+\gamma^2)}, \quad C = 2\alpha^2\frac{2\gamma(\beta-i\gamma)}{(\beta^2+\gamma^2)^2}.$$

Equation (62) has now to be used for evaluating $h_1(\infty;\tau)$ and $l_1(\infty;\tau)$. Since (60) is exactly integrable no further approximations are required. The results are

$$-h_1(\infty;\tau) = \Gamma_1[1-G_0(\tau)]+\Gamma_2[1-G_0(\tau)]^2 + 2\alpha^2\Gamma_3[1-G_0(\tau)]\int_0^\tau [1-G_0(\tau')]\,d\tau', \quad (63)$$

$$-l_1(\infty;\tau) = E_M^{-1}\left\{\exp\left[-\int_0^\tau h_1(\infty;\tau')\,d\tau'\right]\right\} \times\left\{\Gamma_4[1-G_0(\tau)]+\Gamma_5[1-G_0(\tau)]^2+4\alpha^2\Gamma_6[1-G_0(\tau)]\int_0^\tau[1-G_0(\tau')]\,d\tau'\right\}, \quad (64)$$

where

$$\left.\begin{gathered}\Gamma_1 = \frac{2\gamma}{\beta^2+\gamma^2}, \quad \Gamma_2 = -\frac{15\beta^6-23\beta^2-7\gamma^2-\gamma^6}{\beta(\beta^2+\gamma^2)^3(9\beta^2+\gamma^2)}, \\ \Gamma_3 = -\frac{4\gamma^2(3\beta^2-\gamma^2)}{(\beta^2+\gamma^2)^4}, \quad \Gamma_4 = \frac{\beta}{\beta^2+\gamma^2}, \\ \Gamma_5 = \frac{19\beta^4+4+\gamma^4}{\beta(\beta^2+\gamma^2)^3(9\beta^2+\gamma^2)}, \quad \Gamma_6 = -\frac{\beta^2-3\gamma^2}{(\beta^2+\gamma^2)^4}.\end{gathered}\right\} \quad (65)$$

In equations (63) and (64) the coefficients Γ_2 and Γ_5 (Γ_3 and Γ_6) measure the effect of inertial (electromagnetic) nonlinearity within the boundary layers. When (63) and (64) are substituted into (55), we finally obtain at last the following first-order, ordinary, nonlinear, integro-differential equation to be solved for $G_0(\tau)$:

$$\begin{aligned}G_0'(\tau) &= G_0(1-G_0)\left\{\Gamma_1+\Gamma_2(1-G_0)+2\alpha^2\Gamma_3\int_0^\tau[1-G_0(\tau')]\,d\tau'\right\} \\ &+2\alpha^2(1-G_0)\left\{\Gamma_4+\Gamma_5(1-G_0)+4\alpha^2\Gamma_6\int_0^\tau[1-G_0(\tau')]\,d\tau'\right\} \\ &\times\exp\left\{2\int_0^\tau[\Gamma_1(1-G_0)+\Gamma_2(1-G_0)^2]\,d\tau'+2\alpha^2\Gamma_3\left[\int_0^\tau(1-G_0)\,d\tau'\right]^2\right\}, \quad (66)\end{aligned}$$

subject to the initial condition $G_0(0) = \Omega_0/\Omega_1 = 1-\epsilon$.

5.2. *Boundary-layer problem for spin-down*

The initial angular velocity is largest for spin-down; hence $\Omega = \Omega_0$, so

$$\epsilon = (\Omega_1/\Omega_0)-1 \leqslant 0.$$

Since $G_0(0) = 1$ and $G_0(\infty) = 1+\epsilon$, the quantity which measures the deviation of the interior angular velocity from the final value is now $G_0(\tau)-(1+\epsilon)$ and this positive quantity never exceeds $-\epsilon$. A procedure equivalent to that used for spin-up is equally effective here. An expansion for $\tilde{f}$ about conditions at the boundary is introduced:

$$\tilde{f}(\eta;\tau) = i(1+\epsilon)+i[G_0(\tau)-(1+\epsilon)]\tilde{f}^{(1)}(\eta)+i\tilde{f}^{(2)}(\eta;\tau),$$

and then linear problems for $\tilde{f}^{(1)}$ and $\tilde{f}^{(2)}$ are formulated and solved. The result analogous to (62) is

$$\begin{aligned}\tilde{f}(\eta;\tau) &= i(1+\epsilon)+i[G_0(\tau)-(1+\epsilon)]\{1-[1-i\epsilon(\beta+i\gamma)^{-1}\eta]\,e^{-(\beta+i\gamma)\eta}\} \\ &-i[G_0(\tau)-(1+\epsilon)]^2[(A\eta+B)\,e^{-(\beta+i\gamma)\eta}-B\,e^{-2\beta\eta}] \\ &-i[G_0(\tau)-(1+\epsilon)]\left[\int_0^\tau[G_0(\tau')-(1+\epsilon)]\,d\tau'\right]C\eta\,e^{-(\beta+i\gamma)\eta}, \quad (67)\end{aligned}$$

with the same values of A, B and C as before. Equation (67) reduces to (62) in the limit $\epsilon \to 0$, but differs for finite ϵ (this is first evidence of a dissimilarity between spin-up and spin-down – the boundary-layer pump is different). The asymptotic Ekman blowing and Hartmann current at the edge of the boundary layer are found to be

$$h_1(\infty;\tau) = (\Gamma_1+\Gamma_7\epsilon)[G_0(\tau)-(1+\epsilon)]-\Gamma_2[G_0(\tau)-(1+\epsilon)]^2 - 2\alpha^2\Gamma_3[G_0(\tau)-(1+\epsilon)]\int_0^\tau [G_0(\tau')-(1+\epsilon)]\,d\tau', \quad (68)$$

$$l_1(\infty;\tau) = E_M^{-\frac{1}{2}}\left\{\exp\left[-\int_0^\tau h_1(\infty;\tau')\,d\tau'\right]\right\}\left\{(\Gamma_4-\Gamma_8\epsilon)[G_0(\tau)-(1+\epsilon)] - \Gamma_5[G_0(\tau)-(1+\epsilon)]^2 - 4\alpha^2\Gamma_6[G_0(\tau)-(1+\epsilon)]\int_0^\tau [G_0(\tau')-(1+\epsilon)]\,d\tau'\right\}, \quad (69)$$

where
$$\Gamma_7 = \frac{2(\beta^3-3\gamma)}{(\beta^2+\gamma^2)^3}, \quad \Gamma_8 = \frac{3\beta-\gamma^3}{(\beta^2+\gamma^2)^3}. \quad (70)$$

Γ_7 and Γ_8 measure inertial nonlinearity and the others are as before. The final problem for G_0, appropriate for spin-down, found by substituting (68) and (69) into (55), is given by

$$G_0'(\tau) = -G_0(G_0-1-\epsilon)\left[\Gamma_1+\Gamma_7\epsilon-\Gamma_2(G_0-1-\epsilon)-2\alpha^2\Gamma_3\int_0^\tau (G_0-1-\epsilon)\,d\tau'\right] - 2\alpha^2(G_0-1-\epsilon)\left[\Gamma_4-\Gamma_8\epsilon-\Gamma_5(G_0-1-\epsilon)-4\alpha^2\Gamma_6\int_0^\tau (G_0-1-\epsilon)\,d\tau'\right] \times\exp\left\{-2\int_0^\tau [(\Gamma_1+\Gamma_7\epsilon)(G_0-1-\epsilon)-\Gamma_2(G_0-1-\epsilon)^2]d\tau' + 2\alpha^2\Gamma_3\left[\int_0^\tau (G_0-1-\epsilon)\,d\tau'\right]^2\right\}, \quad (71)$$

subject to $G_0(0) = 1$.

6. Solutions and discussion

Some analytical results for certain limits of interest are developed first and then numerical solutions are presented for the general problems contained in (66) and (71).

6.1. *Linearized hydromagnetic flow*

It is readily confirmed that the nonlinear integro-differential equations (66) and (71) reduce properly to the linear problem of Loper & Benton (1970) as $\epsilon \to 0$. For example, in spin-up, $G_0(\tau)$ ranges from $1-\epsilon$ initially to 1 finally, so set $G_0(\tau) = 1-\epsilon G_l(\tau)$ with $G_l(0) = 1$ (the subscript l connoting 'linear'). Substitution into (66), and then division by ϵ, followed by taking the limit $\epsilon \to 0$, shows that G_l satisfies

$$G_l'(\tau) = -(\Gamma_1+2\alpha^2\Gamma_4)\,G_l.$$

In view of the definitions of Γ_1 and Γ_4, the solution yields for the linearized form of $G_0(\tau)$

$$G_0(\tau) \doteq 1-\epsilon\, e^{-\beta\tau},$$

exactly as in Loper & Benton (apart from their slightly different definition of ϵ). A similar linearization of (71) yields

$$G_0(\tau) \doteq 1+\epsilon(1-e^{-\beta\tau}),$$

for spin-down, in agreement with Loper & Benton. Thus, linear hydromagnetic spin-up and spin-down are symmetric processes with the same dimensionless time duration $\tau_s = \beta^{-1} = \gamma$ and this is always shorter than in the hydrodynamic case (γ is a monotonically decreasing function of α). The magnetic field promotes linear spin-up and spin-down through the dominating action of the induced Hartmann current.

6.2. *Nonlinear hydrodynamic spin-up and spin-down*

The influence of inertial and boundary-layer nonlinearity on the ordinary hydrodynamic spin-up problem is studied by setting $\alpha = 0$, which suppresses all MHD effects (especially the electromagnetic nonlinearity). Equations (66) and (71) reduce to

$$G_0'(\tau) = \tfrac{1}{5}G_0(1-G_0)(6-G_0), \quad G_0(0) = 1-\epsilon \quad \text{for spin-up,} \tag{72}$$

$$G_0'(\tau) = \tfrac{1}{5}G_0(1-G_0+\epsilon)(6-G_0-\tfrac{3}{2}\epsilon), \quad G_0(0) = 1 \quad \text{for spin-down.} \tag{73}$$

Clearly spin-up is no longer simply a symmetric reflexion of spin-down. Solutions of these equations, in implicit form, are

$$\tau = \ln\left[\left(\frac{G_0}{1-\epsilon}\right)^{\frac{5}{6}}\left(\frac{6-G_0}{5+\epsilon}\right)^{\frac{1}{6}}\left(\frac{\epsilon}{1-G_0}\right)\right] \quad \text{for spin-up,} \tag{74}$$

$$\tau = \ln\left[G_0^{10/3(1+\epsilon)(4-\epsilon)}\left(\frac{1+\epsilon-G_0}{\epsilon}\right)^{-2/(1+\epsilon)(2-\epsilon)}\left(\frac{12-3\epsilon-2G_0}{10-3\epsilon}\right)^{4/3(2-\epsilon)(4-\epsilon)}\right] \quad \text{for spin-down.} \tag{75}$$

It is of interest to compare these formulae with the analogous results from the work of Greenspan & Weinbaum (1965). In the present notation their approximate solutions are

$$\tau = \tfrac{4}{5}(1-\epsilon)^{-\frac{3}{2}}\{G_0-1+\epsilon-\tfrac{5}{8}(2-3\epsilon)\ln[(1-G_0)/\epsilon]\} \quad \text{for spin-up,} \tag{76}$$

$$\tau = \tfrac{4}{5}(G_0-1)-(1-\tfrac{1}{2}\epsilon)\ln[(1+\epsilon-G_0)/\epsilon] \quad \text{for spin-down.} \tag{77}$$

Numerical comparison of (74) and (75) with (76) and (77) reveals them to be virtually equivalent for $|\epsilon|$ up to $\frac{1}{3}$ and only slight (of order less than 15 %) departures are present for $|\epsilon| = 0{\cdot}5$. At larger values of $|\epsilon|$ the Greenspan–Weinbaum procedure cannot be a valid approximation because of neglected higher order terms in ϵ. Although Greenspan (1968, p. 169) notes that their method cannot be used for spin-up from rest ($\epsilon = +1$), equation (76) (or equation (59) of the paper by Greenspan & Weinbaum) shows that lack of validity actually sets in for smaller ϵ: equation (76) becomes useless at $\epsilon = \frac{2}{3}$, which corresponds to $\epsilon = 2$ in the notation of Greenspan & Weinbaum. A subsidiary purpose of the present work then has been to rectify this deficiency. In contrast, note that either (72) or (74) predicts $G_0(\tau) = 0$ for the case of spin-up from rest ($\epsilon = +1$). The implied stagnant interior flow is clearly correct physically in a

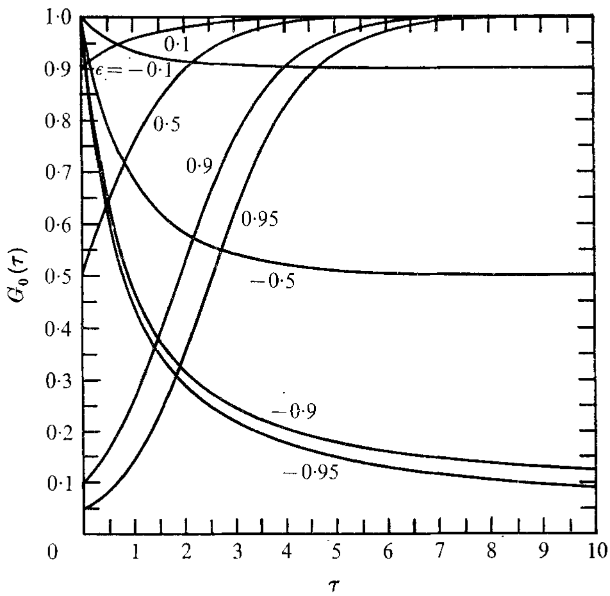

FIGURE 2. Normalized interior angular velocity G_0 versus dimensionless time τ for nonlinear ordinary hydrodynamic spin-up and spin-down between identical pairs of angular velocities.

laterally unbounded container because, on the Ekman spin-up time scale, there is, in that case, no interior angular momentum to be convected and there are no interior vortex lines to be stretched. Moreover, the work of Wedemeyer (1964) (described and verified qualitatively by Greenspan 1968) shows that a non-rotating core is still present on the Ekman spin-up time scale when the fluid is confined laterally by cylindrical side walls. This serves again to emphasize that, in applying the present work to fully contained fluids, one must restrict attention to the evolution of the interior angular velocity at or near the axis of rotation and for times only to order $\Omega^{-1}E^{-\frac{1}{2}}$ (not the longer viscous diffusion time $\Omega^{-1}E^{-1}$).

The solutions in (74) and (75) are plotted as functions of τ for $\epsilon = \pm 0{\cdot}1$, $\pm 0{\cdot}5$, $\pm 0{\cdot}9$ and $\pm 0{\cdot}95$ in figure 2. Recall that G_0 is the interior angular velocity normalized by the larger of the initial (Ω_0) and final (Ω_1) angular velocities and τ is the time normalized by the ordinary linear Ekman spin-up time (using the larger of Ω_0 and Ω_1). The pair of curves associated with the same magnitude of Rossby number describe spin-up and spin-down between the same two values of angular velocity. For $\epsilon = \pm 0{\cdot}1$, linear theory is adequate and the fluid response is purely exponential in time with spin-up time equal to spin-down time. However, figure 2 reveals that, for larger $|\epsilon|$, spin-down from Ω_0 to Ω_1 takes significantly longer than spin-up from Ω_0 to Ω_1. Thus, for $\epsilon = 0{\cdot}9$, G_0 has approached its spin-up asymptote by $\tau = 7$, whereas the spin-down asymptote for $\epsilon = -0{\cdot}9$ is not yet closely approached even at $\tau = 10$ and the final approach is slow; e.g., at $\tau = 30$ for $\epsilon = -0{\cdot}9$, $G_0 = 0{\cdot}10122$. In fact, the ultimate decay in G_0 is predicted to be algebraic in time rather than exponential, for large negative ϵ (e.g. the asymptotic form of (75) for $\epsilon = -1$, spin-down to rest, is $G_0 \sim (1+\frac{3}{2}\tau)^{-1}$). Furthermore, despite an evident general increase with $|\epsilon|$ in initial angular

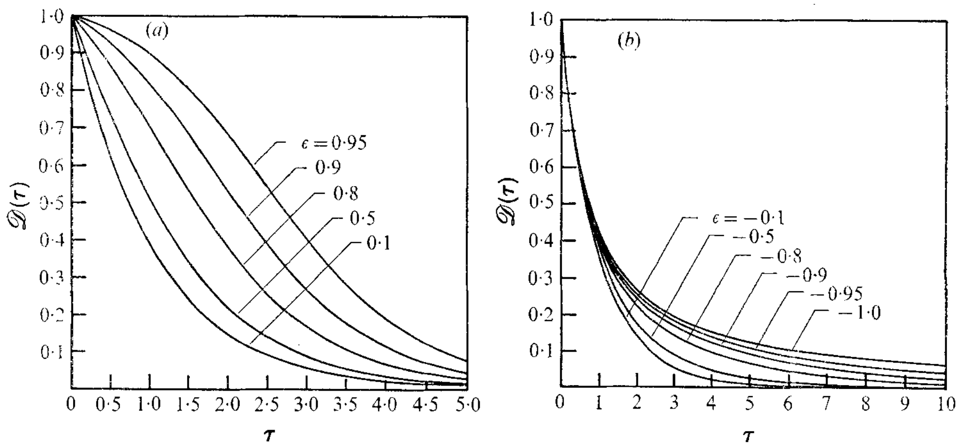

FIGURE 3. Normalized instantaneous deviation $\mathscr{D}$ of angular velocity from the final value versus dimensionless time τ in (a) hydrodynamic spin-up and (b) hydrodynamic spin-down, for various Rossby numbers ϵ.

acceleration or deceleration both spin-up and spin-down require progressively more time to be completed as the nonlinearity increases. This is at least partially due to our use of the larger of Ω_0 and Ω_1 for scaling purposes. For linear flow there is little difference between the two, but for larger $|\epsilon|$, obviously Ω is not truly representative of the whole transient process, being in the sense of an over-estimate of the characteristic angular speed.

Further comparison of spin-up and spin-down is facilitated by considering a quantity which is the instantaneous deviation in angular velocity from the final value normalized by the initial deviation from the final value:

$$\mathscr{D}(\tau) \equiv [G_0(\tau) - G_0(\infty)]/[G_0(0) - G_0(\infty)]. \tag{78}$$

Clearly, $\mathscr{D}(0) = 1$ and $\mathscr{D}(\infty) = 0$ independently of ϵ. For example, the linearized hydromagnetic solutions above for both spin-up and spin-down assume the same form, $\mathscr{D}_l(\tau) = e^{-\beta\tau}$.

Figures 3(a) and (b) are plots for spin-up and spin-down, respectively, of $\mathscr{D}(\tau)$ versus τ for various values of ϵ. It is seen that spin-up is relatively sensitive to Rossby number but spin-down, at least in the initial phases, is but weakly dependent on ϵ. Partial spin-up is progressively slower as the nonlinearities increase, but by time $\tau = 5$ spin-up is essentially complete for all ϵ (except, of course, $\epsilon = +1$), whereas strongly nonlinear spin-down is still not complete at that time.

The behaviour displayed in these figures is the result of both boundary-layer and inertial nonlinearity and these can either co-operate or oppose each other. Furthermore, as noted previously the boundary layers are most nonlinear initially whereas the inertial nonlinearity is then absent. Consequently, for example, in spin-down (figure 3(b)) the effect of inertial nonlinearity in the interior is such as to reduce the rate of spin-down in the later stages, but, as seen in Greenspan (1968), a nonlinear boundary layer on a surface in spin-down pumps more vigorously than its linear counterpart so this tends to promote nonlinear spin-down in the early stages. Figure 3 (*b*) then implies that boundary-layer and inertial nonlinearity effectively counterbalance each other until times of order $\tau = 1$, after which inertial nonlinearity dominates. In contrast, Ekman layers on boundaries being spun up pump at most only slightly more vigorously than in linear theory (see Benton & Chow 1972) so the sluggishness associated with finite inertia of the interior fluid always dominates and nonlinear spin-up is retarded for all τ (refer to figure 3 (*a*)).

The cross-over behaviour present for both spin-up and spin-down in Greenspan's figure 3.16 but clearly absent for spin-up here is due, at least in part, to his use of the initial angular speed for normalization. This coincides with our Ω for spin-down but is an underestimate for spin-up (this also partially accounts for the failure of the Greenspan–Weinbaum theory for strongly nonlinear spin-up – their normalizing initial angular velocity is tending to zero in that limit). Incidentally, the favourable comparison of Greenspan & Weinbaum's theory with Pearson's exact numerical computations may in part be fortuitous. In the present notation, Pearson's Ekman number gives $E^{\frac{1}{2}} = 0{\cdot}063$, which is not particularly small. Although it is difficult to correct the present asymptotic theory for the effects of finite Ekman number, it is none the less clear that there are at least two modifications which compete with each other. For finite $E^{\frac{1}{2}}$, the boundary layers are not vanishingly thin so there is less inviscid fluid to be spun up or spun down. On the other hand, the boundary-layer formation time, of order $t \sim \Omega^{-1}$ or $\tau \sim E^{\frac{1}{2}}$, is also no longer negligible and this acts to delay spin-up or spin-down. Indeed, crude calculations suggest that the theoretical curves in figures 3 (*a*) and (*b*) can be shifted either to the right or left, depending on circumstances, if finite E effects are included. For this reason it would be useful to rerun the Pearson calculation at smaller E.

Generally, the present theory agrees adequately with that of Greenspan & Weinbaum where both are valid but extends the range of allowable $|\epsilon|$ to the complete range $-1 \leqslant \epsilon \leqslant 1$.

6.3. *The electromagnetic nonlinearity*

The influence of the axial Ekman secondary flow in stretching or compressing the basic axial magnetic field lines is referred to as the electromagnetic nonlinearity. A convenient gross measure of its strength is provided by a quantity, $\mathscr{E}(\alpha, \epsilon)$, say (with $\mathscr{E}$ for 'electromagnetic') defined as the magnitude of the net fractional change in axial magnetic field between the initial and final states:

$$\begin{aligned}\mathscr{E}(\alpha,\epsilon) &\equiv \left|\frac{B_z(\tau\to\infty)-B_z(\tau\to 0)}{B_z(\tau\to 0)}\right| \\ &= |M_0(\infty)-1| = \left|\exp\left(-\int_0^\infty h_1(\infty;\tau)\,d\tau\right)-1\right|, \qquad (79)\end{aligned}$$

where (47) and (54) have been used. The axial velocity function $h_1(\infty, \tau)$ needed here is given in terms of $G_0(\tau)$ in (63) for spin-up and in (68) for spin-down. Figure 4 displays $\mathscr{E}(\alpha, \epsilon)$ versus ϵ for several values of α, where values of $G_0(\tau)$ from numerical integration have been used. Except for $\alpha = 0$ and $\epsilon \geqslant 0{\cdot}5$, the axial field undergoes fractional changes of at most 100 %. The general increase in $\mathscr{E}$

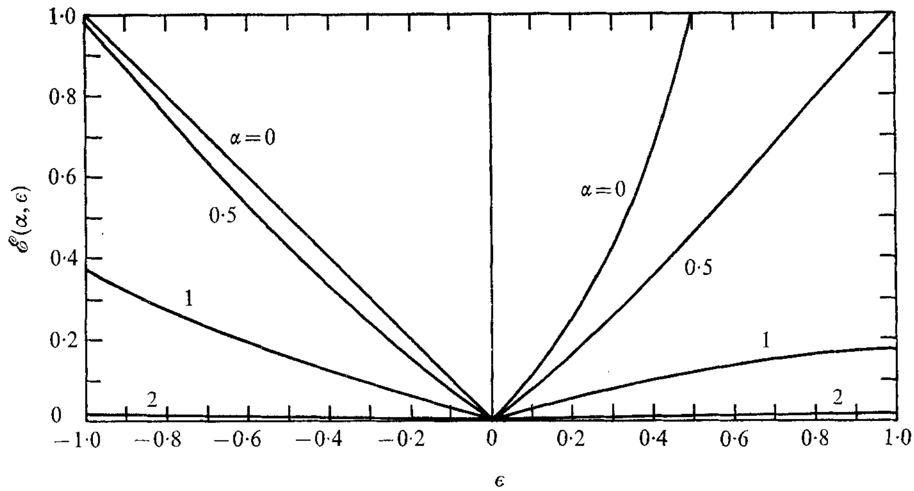

FIGURE 4. Strength of the electromagnetic nonlinearity $\mathscr{E}$ as a function of Rossby number ϵ for several values of magnetic interaction parameter α.

with $|\epsilon|$ for fixed α is due to the ordinary increase in Ekman pumping with Rossby number. The apparent decrease in electromagnetic nonlinearity with increasing field strength (i.e. α) occurs because the Ekman pumping is increasingly suppressed by a strong field. The curve for $\alpha = 0$ rises towards infinity as $\epsilon \to 1$ because, for spin-up from rest, Ekman suction continuously stretches the field lines but spin-up of the interior fluid does not occur (on the time scale of this analysis). When it is recalled that linearized theory allows no fractional change in B_z (so $\mathscr{E} = 0$) then we must conclude that in general substantial ultimate changes in the applied magnetic field can be effected by even a slow secondary flow; the electromagnetic nonlinearity is important.

6.4. *Strongly magnetic flow*

The conclusion just reached prompts investigation of the transient effects of strong electromagnetic nonlinearity isolated from other complications. It is easily seen from (55) that, as α increases beyond 1, the evolution of interior angular velocity depends more and more on the Lorentz torque and less and less on the nonmagnetic mechanism (conservation of angular momentum). For large α the boundary-layer nonlinearity present in $h_1(\infty;\tau)$ and $l_1(\infty;\tau)$ is also suppressed (cf. Benton & Chow 1972). Consequently, sensible approximations to (66) and (71) for magnetically dominated flow are

$$G_0'(\tau) \doteq 2\alpha^2\Gamma_4(1-G_0)\exp\left\{2\Gamma_1\int_0^\tau [1-G_0(\tau')]\,d\tau'\right\} \quad \text{for spin-up;}$$

$$G_0'(\tau) \doteq -2\alpha^2\Gamma_4(G_0-1-\epsilon)\exp\left\{-2\Gamma_1\int_0^\tau [G_0(\tau')-1-\epsilon]\,d\tau'\right\} \quad \text{for spin-down.}$$

These equations can be immediately integrated once and the exact solutions which satisfy the initial conditions are identical in form when written in terms of $\mathscr{D}$ rather than G_0 (cf. equation (78)):

$$\mathscr{D}(\tau) = (\alpha^2\beta^2+2\epsilon)\{\alpha^2\beta^2\exp[\Gamma_1(\alpha^2\beta^2+2\epsilon)\tau]+2\epsilon\}^{-1}. \tag{80}$$

This solution reveals that the dimensionless spin-up or spin-down time τ_s for strongly magnetic flow is

$$\tau_s = (\beta^2+\gamma^2)/2\gamma(\alpha^2\beta^2+2\epsilon). \tag{81}$$

Since ϵ is positive (negative) for spin-up (spin-down) we see that, as for non-magnetic flow, spin-down takes longer than spin-up. However, the asymmetry here is not large because in order for (80) to be a valid approximation we require a strong magnetic field, say $\alpha \geqslant 2$ (based on the forms of $\beta(\alpha)$, $\gamma(\alpha)$ and the results of linear theory), and in this situation $\alpha^2\beta^2 \geqslant 31$ but $|2\epsilon|$ is at most 2; the Rossby number effect is limited to of order $\pm 7\%$. Moreover, when $\alpha \geqslant 2$, then $\beta^2 \gg \gamma^2$ so τ_s above reduces very nearly to γ, as for linearized hydromagnetic flow. This analysis then shows that only for a moderate range of imposed magnetic fields $\alpha \leqslant 2$ is nonlinearity important. Also, the quantitative enhancement of both spin-up and spin-down by a strong field (compared to non-magnetic flow) is given by (80) and (81).

6.5. *Numerical solutions*

The first-order, ordinary, nonlinear, integro–differential equations (66) and (71) are readily solved numerically under the given initial conditions as marching problems. The integrals were evaluated by the trapezoid rule and a step size of $\Delta\tau = 0{\cdot}001$ was found to be adequate. Two independent checks (in addition to mesh refinement studies) are possible. Upon equating α to 0 we find excellent agreement between the numerical results and the analytic solutions in (74) and (75). Also, calculations at large α give the same results as those found from the analytic solution in (80).

Figure 5 displays numerical results for spin-up and spin-down between the same sets of values of Ω_0 and Ω_1 but with a modest ($\alpha = 1$) magnetic field present. Comparison with figure 2 (noting the different abscissa scale factor) reveals a general decrease in time duration of both spin-up and spin-down as well as a weakened asymmetry between the two processes.

Figures 6 (*a*) and (*b*) give results for $\mathscr{D}(\tau)$ similar to those in figures 3(*a*) and (*b*) but with ϵ held fixed (at $\pm 0{\cdot}95$) and α variable. The dashed curves here give, for comparison, the theory for linear hydrodynamic spin-up and spin-down. Two important conclusions emerge from these results. First, for fixed Rossby number an increasing magnetic field always promotes both spin-up and spin-down but a weak field (say $\alpha \leqslant 0{\cdot}5$) is relatively ineffective for spin-down, yet highly effective for spin-up, the trend being most pronounced at large $|\epsilon|$. The explanation of this clear asymmetry between the two processes is readily understandable in terms of the electromagnetic nonlinearity. During spin-up, the axial magnetic field continually intensifies because of field line stretching so even a weak initial field is subsequently effective (recall figure 4 for $\epsilon > 0$ and small α). However, axial field lines are continually dispersed laterally during spin-down so the magnetic coupling is less effective and an initially weak field becomes even more feeble in time.

Another conclusion confirmed by these figures is that a sufficiently strong magnetic field, say $\alpha \geqslant 2$, suppresses any difference between spin-up and spin-down, and produces a flow substantially independent of Rossby number (note

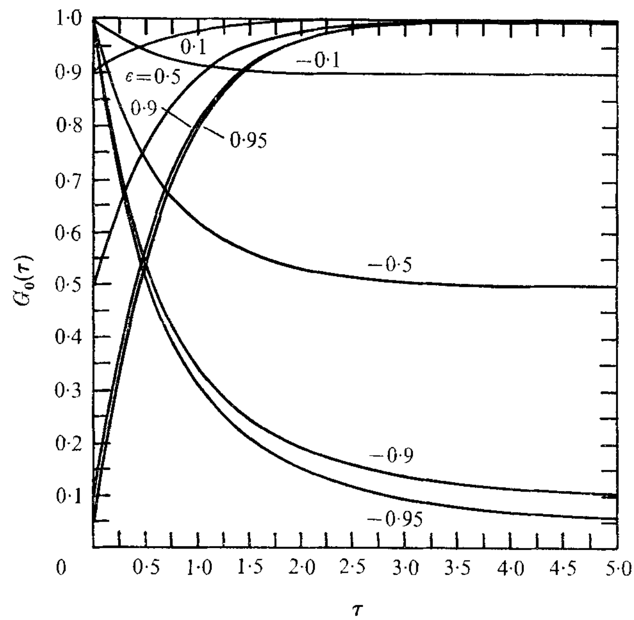

FIGURE 5. Normalized interior angular velocity G_0 versus dimensionless time τ for hydromagnetic ($\alpha = 1$) spin-up and spin-down between identical pairs of angular velocities.

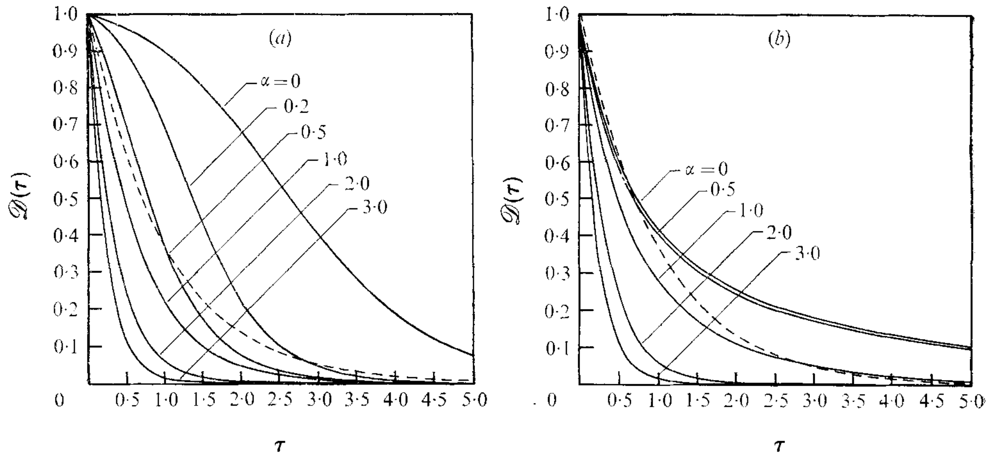

FIGURE 6. Normalized instantaneous deviation $\mathscr{D}$ of angular velocity from the final value versus dimensionless time τ in strongly nonlinear (*a*) spin-up ($\epsilon = 0{\cdot}95$) and (*b*) spin-down ($\epsilon = -0{\cdot}95$) for several values of magnetic interaction parameter α. ---, $e^{-\tau}$.

the similarity of profiles for $\alpha \geqslant 2$ in figure 6). This again suggests that a strong field dominates all nonlinearities. As shown by Benton & Chow (1972), the Ekman–Hartmann pump becomes linear in this situation (because secondary flow is suppressed by a strong field). Furthermore, weakened Ekman pumping leads to only a minute electromagnetic nonlinearity (cf. figure 4 for large α). The inertial nonlinearity in the fluid interior survives, but in the strongly magnetic limit this is dominated by the electromagnetic body torque as a spin-up or spin-down mechanism.

The flow studied in this paper does not, of course, constitute a dynamo but it shares certain features. The imposed axial field corresponds to a poloidal field in spherical co-ordinates. As for dynamos such a field is stretched out into a toroidal field by the differential rotation across the boundary layers. The crucial half of a dynamo cycle, namely the regeneration of poloidal field from toroidal field, is impossible in the present flow because of the assumed axisymmetry. None the less, the imposed poloidal field does not decay away, despite the presence of both ohmic and viscous dissipation. In fact, during spin-up such a field is replenished by stretching of the initial poloidal field lines and radially inward convection (by secondary flow) of poloidal fields. In essence, the present flow fails to constitute a dynamo because the field is maintained laterally at infinity.

The present theory pays no attention to the important question of hydrodynamic instability. All that can be said at this point is that, for the laterally unbounded case treated herein, instability of the Ekman–Hartmann layers is the major potential problem, but Gilman (1971) has shown such layers to be much less unstable than the pure Ekman layer.

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